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Phase Correction of a Laser Beam by Gas Jets

E. Wasserstrom* and W. H. Christiansen†
University of Washington, Seattle, Wash.

When a laser beam propagates in the atmosphere, its phase can be distorted either passively, due to turbulence, or actively, due to thermal blooming. It has been demonstrated that it is possible to partially compensate for these phase errors by physically distorting the transmitting optics. Here one aspect of another method is studied where the correcting active elements are gas jets of controllable density variation that are suitably interposed in the beam. The degree of phase compensation, which may be achieved in principle, has been studied using ideal jets. It is shown that for the thermal blooming case the mean square phase error can be reduced by nearly 50% with a system of 7 × 7 crossed jets.

I. Introduction

THE problem dealt with herein was encountered while evaluating a new concept of adaptive optics for laser beams, that was proposed several years ago.[1] The purpose of adaptive optics is to be able to correct the phase distortion of a laser beam in real time. This is usually achieved[2,3] by physically distorting the transmitting optics, by deforming flexible mirrors, and thus varying the optical lengths of the various rays of the beam.

In the new method, the active elements in the transmitting optics are replaced by gas jets of controllable density that are suitably interposed in the beam. Since the index of refraction affects the speed of light in a gas, active control of the density field of the jets in turn adjusts the phase distortion of the beam as it propagates through the medium. In order to illustrate the concept, a uniformly illuminated laser beam propagating in the z direction is shown in Fig. 1. Suppose that the phase distribution over the laser beam is originally $\phi(x,y)$. The reduction in peak light intensity relative to the beam is given by[4]

$$I = I_0 [1 - \overline{(\Delta\phi)^2}] \tag{1}$$

where I_0 is the intensity for a perfect (unaberrated) beam, and $\overline{\Delta\phi^2}$ is the mean square error of phase over the beam cross section (after subtraction of tilt and focus). In deriving Eq. (1) it has been assumed that the aberrations are small, say $\overline{\Delta\phi^2} \lesssim (\pi/5)^2$. In order to reduce this phase distortion (and thus increase the maximum light intensity), gas jets are passed through the beam in two orthogonal directions (see Fig. 1). If the density variation of the ideal jet (defined as a jet in which diffusion and entrainment are absent) in the y direction is denoted by $\rho(x,z)$, then the relative phase change of the laser beam traversing the jet is given by[5]

$$f(x) = -\frac{2\pi\beta}{\lambda} \int_0^t [\rho(x,z) - \rho_r] \, dz \tag{2}$$

where λ is the wavelength of the laser light, t the thickness of the jet, ρ_r some reference density, and β the Gladstone-Dale constant. Similarly, the phase change due to the jet in the x direction is denoted by $g(y)$. Since the phase changes due to the two jets are additive, the phase of the laser beam after passing the two jets is

$$\psi(x,y) = \phi(x,y) - f(x) - g(y) \tag{3}$$

The main problem addressed in this jpaper is the determination of the functions $f(x)$ and $g(y)$ such that $\overline{\psi^2}$, the mean square of the new phase function, is minimized. It is recognized that real jets will have an effect on the performance of the fluid adaptive optics scheme since jets can cause optical abberations of their own. Thus, what is studied here is the ideal or best phase compensation that may be achieved. In the next section a discrete case is studied and in the following section the continuous case is treated. In Sec. IV numerical examples are solved.

II. Discrete Case

In this section the cross section of the laser beam is assumed to be a rectangle D. The phase correction is performed by M constant density jets in the x direction and N constant density jets in the y direction (see Fig. 2). The intersection of the boundaries of these jets subdivide D into $M \times N$ smaller rectangles in each of which the phase ϕ_{ij} is assumed constant.

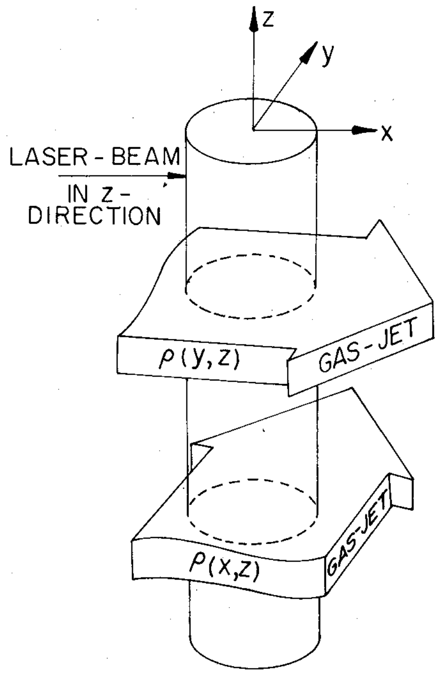

Fig. 1 Concept of the method of laser beam control by crossed jets.

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*Associate Professor, on leave from Technion, Israel Institute of Technology, Haifa, Israel.

†Professor, Department of Aeronautics and Astronautics. Member AIAA.

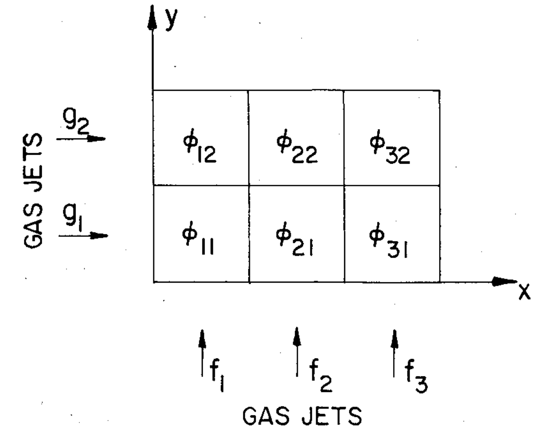

Fig. 2 Projection of discrete crossed jets with a rectangular domain.

The problem is to determine f_i and g_j so that

$$\psi_{ij} = \phi_{ij} - f_i - g_j, \qquad i = 1,2,\ldots,M, \qquad j = 1,2,\ldots,N \tag{4}$$

minimizes J where

$$J = \sum_{i,j} A_{ij}(\phi_{ij} - f_i - g_j)^2 \tag{5}$$

where the A_{ij} are the areas of the rectangles.

To minimize J we must have $\partial J/\partial f_i = \partial J/\partial g_j = 0$, i.e.,

$$Nf_i + \sum_{j=1}^{N} g_j - \sum_{j=1}^{N} \phi_{ij} = 0, \qquad i = 1,2,\ldots,M \tag{6}$$

$$Mg_j + \sum_{i=1}^{M} f_i - \sum_{i=1}^{M} \phi_{ij} = 0, \qquad j = 1,2,\ldots,N \tag{7}$$

where all A_{ij} were assumed to be equal. Thus, the analysis is restricted to equal width jets. Jets of different widths are a possibility, but have not been considered in this or in subsequent calculations.

The $M \times N$ equations [Eqs. (6) and (7)] for the unknowns f_i and g_j have a nonunique solution, since, if f_i and g_j is a solution, so is $f_i + C$, $g_j - C$, where C is an arbitrary constant. In order to make the solution unique we require the minimization of the function

$$p(f_i, g_j) = \sum_{i=1}^{M} f_i^2 + \sum_{j=1}^{N} g_j^2 \tag{8}$$

subject to the constraints given by Eqs. (6) and (7).

This problem is solved by the method of Lagrange multipliers by defining

$$S = \sum_i f_i^2 + \sum_j g_j^2 + \sum_i \lambda_i \Big(Nf_i + \sum_j g_j - \sum_j \phi_{ij} \Big) + \sum_j \mu_j \Big(Mg_j + \sum_i f_i - \sum_i \phi_{ij} \Big) \tag{9}$$

where λ_i and μ_j are the Lagrange multipliers. Hence,

$$\frac{\partial S}{\partial f_i} = 2f_i + N\lambda_i + \sum_j \mu_j = 0 \tag{10}$$

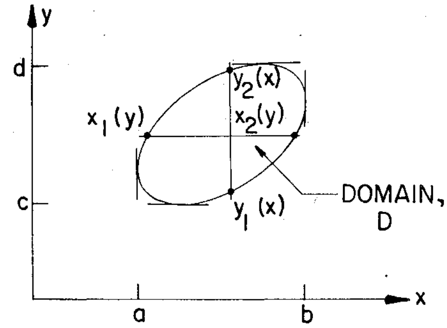

Fig. 3 The generalized domain using continuous freejets.

$$\frac{\partial S}{\partial g_j} = 2g_j + \sum_i \lambda_i + M\mu_j = 0 \tag{11}$$

Solving Eqs. (6, 7, 10 and 11) for f_i, g_j, λ_i and μ_j, one obtains

$$f_i = \frac{1}{N}\Big(\sum_{j=1}^{N} \phi_{ij} - \bar{\phi} \Big), \qquad i = 1,2,\ldots,M \tag{12}$$

$$g_j = \frac{1}{M}\Big(\sum_{i=1}^{M} \phi_{ij} - \bar{\phi} \Big), \qquad j = 1,2,\ldots,N \tag{13}$$

where

$$\bar{\phi} \equiv \frac{1}{M+N} \sum_{j=1}^{N} \sum_{i=1}^{M} \phi_{ij} = \sum_{i=1}^{M} f_i = \sum_{j=1}^{N} g_j \tag{14}$$

III. Continuous Case

Here the cross section of the laser beam is assumed to be a convex domain D in which its phase error $\phi(x,y)$ is continuously given (see Fig. 3). This phase is to be decreased by two crossed jets of variable index of refraction in the x and y directions, as described in the introduction, so that the mean square of the new phase ψ, defined in Eq. (3), is minimized, i.e.,

$$J = \int\int_D [\phi(x,y) - f(x) - g(y)]^2 \mathrm{d}x\mathrm{d}y = \min \tag{15}$$

Equation (15) is solved by standard variational methods.[6] Varying $f(x)$ to $f(x) + \epsilon\xi(x)$ and using the condition that $[\mathrm{d}J(\epsilon)/\mathrm{d}\epsilon]_{\epsilon=0} = 0$, one obtains

$$[y_2(x) - y_1(x)]f(x) + \int_{y_1(x)}^{y_2(x)} g(y)\mathrm{d}y - \int_{y_1(x)}^{y_2(x)} \phi(x,y)\mathrm{d}y = 0 \tag{16}$$

Similarly, varying $g(y)$ yields

$$\int_{x_1(y)}^{x_2(y)} f(x)\mathrm{d}x + [x_2(y) - x_1(y)]g(y) - \int_{x_1(y)}^{x_2(y)} \phi(x,y)\mathrm{d}x = 0 \tag{17}$$

Again, the solution of these two integral equations is not unique, and to make it so we require the minimization of the functional

$$p(f,g) = \int_a^b [f(x)]^2\mathrm{d}x + \int_c^d [g(y)]^2\mathrm{d}y \tag{18}$$

As in the previous section, the problem is solved by using Lagrange multipliers $\lambda(x)$ and $\mu(y)$. After some manipulations one can show that

$$\int_a^b f(x)\mathrm{d}x = \int_c^d g(y)\mathrm{d}y \equiv \bar{\phi} \tag{19}$$

For the special case where the domain D in Fig. 3 is a rectangle, the equation can be solved explicitly to yield

$$f(x) = \frac{1}{d-c}\left[\int_c^d \phi(x,y)\mathrm{d}y - \bar{\phi}\right] \tag{20}$$

$$g(y) = \frac{1}{b-a}\left[\int_a^b \phi(x,y)\mathrm{d}x - \bar{\phi}\right] \tag{21}$$

where

$$\bar{\phi} = \frac{1}{(b-a)+(d-c)}\int_c^d\int_a^b \phi(x,y)\mathrm{d}x\mathrm{d}y \tag{22}$$

which is similar to the solution obtained previously in Eqs. (12-14).

Unfortunately, it is not possible to write an explicit solution for the general case, where the domain is not a rectangle. One way to proceed is to solve Eqs. (16) and (17) iteratively, as follows:

$$f^{(n)}(x) = \frac{1}{y_2(x)-y_1(x)}\int_{y_1(x)}^{y_2(x)} \phi(x,y)\mathrm{d}y - \frac{1}{y_2(x)-y_1(x)}\int_{y_1(x)}^{y_2(x)} g^{(n-1)}(y)\mathrm{d}y \tag{23}$$

$$g^{(n)}(y) = \frac{1}{x_2(y)-x_1(y)}\int_{x_1(y)}^{x_2(y)} \phi(x,y)\mathrm{d}x - \frac{1}{x_2(y)-x_1(y)}\int_{x_1(y)}^{x_2(y)} f^{(n)}(x)\mathrm{d}x \tag{24}$$

where $n=1,2\ldots$ counts the iterations and $f^{(0)}(x)$ and $g^{(0)}(y)$ are guessed. This method has been used successfully for the case of the circular aperture and the results will be described in the next section. Another way to proceed is to decouple Eqs. (23) and (24), but this method will not be detailed here.

IV. Numerical Example

When a laser beam propagates in the atmosphere, it heats the air and consequently is distorted. This phenomenon is called thermal blooming. The phase distortion of a circular beam has been predicted by Bradley and Hermann.[7] Recently Bushnell and Skogh[8] have computed the phase compensation that can be achieved for a thermally bloomed laser beam by mirror deformation. In what follows a similar problem will be solved, except that now the compensation is achieved by means of gas jets as described earlier.

The phase distortion of a thermally bloomed beam is giver in polar coordinates by[7,8]

$$\phi(r,\theta) = \sum_{n=0}^{3} f_n(r)\cos n(\theta-\theta_0) \tag{25}$$

where $r=\sqrt{x^2+y^2}\leq 1$, $\theta=\tan^{-1}(x/y)$ and θ_0 is the direction of the wind. The functions $f_n(r)$ are given by

$$\begin{aligned}
f_0 &= C_1[A_{20}z(4)+A_{40}z(11)+A_{60}z(22)]\\
f_1 &= C_1[A_{11}z(2)+A_{31}z(7)+A_{51}z(16)]\\
f_2 &= C_1[A_{22}z(5)+A_{42}z(12)]\\
f_3 &= C_1[A_{33}z(9)]
\end{aligned} \tag{26}$$

where C_1 is a nondimensional coefficient equal to ¾ in the calculations. This coefficient is equal to the constant C [Eq. (11) of Ref. 8] divided by the wave number of CO_2 laser radiation. The Zernike polynomials $z(i)$ in Eq. (26) are defined by[4]

$$\begin{aligned}
&z(2)=r, \quad z(4)=2r^2-1, \quad z(5)=r^2\\
&z(7)=3r^3-2r, \quad z(9)=r^3, \quad z(11)=6r^4-6r^2+1\\
&z(12)=4r^4-3r^2, \quad z(16)=10r^5-12r^3+3r\\
&z(22)=20r^6-30r^4+12r^2-1
\end{aligned} \tag{27}$$

Finally, the coefficients A_{ij} in Eq. (26) are given in Ref. 8 as

$$\begin{aligned}
&A_{20}=-0.053 \quad A_{40}=-0.050 \quad A_{60}=+0.021\\
&A_{11}=+0.440 \quad A_{31}=+0.072 \quad A_{51}=-0.061\\
&A_{22}=+0.101 \quad A_{42}=+0.096 \quad A_{33}=-0.075
\end{aligned} \tag{28}$$

The first exercise considered was to reduce the phase error $\phi(r,\theta)$ defined in Eq. (25) by N equal width jets in the x direction and N equal width jets in the y direction (see Fig. 4). The solution obtained in Sec. II, i.e., Eqs. (12-14), cannot be applied in a straightforward manner since the domain D is now circular rather than rectangular, and, also, the phase is not constant over the subrectangles A_{ij} generated by the intersection of the jets. The problem was solved by two methods. In the first method the functional J in Eq. (15) was rewritten as

$$J_N(f_i,g_j)\sum_{i,j=1}^{N}\int_{A_{ij}}[\phi(x,y)-f_i-g_j]^2\mathrm{d}x\mathrm{d}y \tag{29}$$

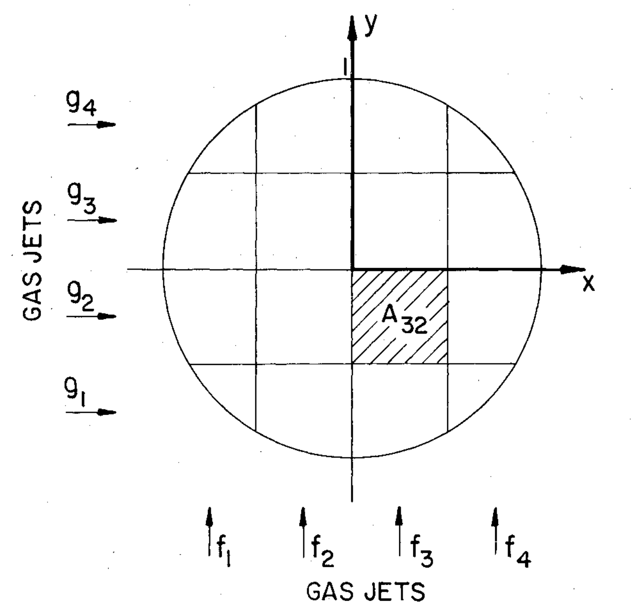

Fig. 4 A circular laser beam covered by discrete jets.

Table 1 Exact numerical results for $N \times N$ jets with $\theta_0 = 0$

N[a] / i	1		2		3		4		5		6		7	
	f_i	g_j	f_i	g_j	f_i	g_j	f_i	g_j	f_i	g_j	f_i	g_j	f_i	g_j
1	0	0	0	−0.184	−0.015	−0.267	−0.039	−0.316	−0.064	−0.343	−0.086	−0.360	−0.104	−0.366
2	$J_1 = 0.1702$		0	0.184	0.026	−0.007	0.032	−0.107	0.036	−0.176	0.034	−0.230	0.026	−0.273
3			$J_2 = 0.0631$		−0.015	0.271	0.032	0.094	0.026	−0.004	0.032	−0.070	0.039	−0.122
4					$J_3 = 0.0353$		−0.039	0.313	0.036	0.161	0.032	0.061	0.028	−0.003
5							$J_4 = 0.0248$		−0.064	0.336	0.034	0.205	0.039	0.111
6									$J_5 = 0.0189$		−0.086	0.350	0.026	0.236
7											$J_6 = 0.0154$		−0.104	0.365
													$J_7 = 0.0131$	

[a] N varies from 1 to 7.

Table 2 Exact numerical results for $N \times N$ jets with $\theta_0 = \pi/4$

N[a] / i	1		2		3		4		5	
	f_i	g_j	f_i	g_j	f_i	g_j	f_i	g_j	f_i	g_j
1	0	0	−0.121	−0.121	−0.193	−0.193	−0.250	−0.250	−0.293	−0.293
2	$J_1 = 0.1702$		0.121	0.121	0.010	0.010	−0.038	−0.039	−0.079	−0.079
3			$J_2 = 0.0774$		0.180	0.180	0.065	0.064	0.011	0.011
4					$J_3 = 0.0426$		0.210	0.209	0.108	0.108
5							$J_4 = 0.0286$		0.225	0.224
									$J_5 = 0.0221$	

[a] N varies from 1 to 5.

Table 3 Approximated numerical results for $N \times N$ jets with $\theta_0 = 0$

N[a] / i	1		2		3		4		5	
	f_i	g_j	f_i	g_j	f_i	g_j	f_i	g_j	f_i	g_j
1	0	0	0	−0.184	−0.020	−0.272	−0.040	−0.322	−0.064	−0.350
2	$J_1 = 0.1695$		0	0.184	0.030	−0.010	0.033	−0.109	0.036	−0.182
3			$J_2 = 0.0627$		−0.020	0.273	0.033	0.096	0.028	−0.006
4					$J_3 = 0.0392$		−0.040	0.319	0.036	0.166
5							$J_4 = 0.0252$		−0.064	0.344
									$J_5 = 0.0196$	

[a] N varies from 1 to 5.

Table 4 Approximated numerical results for $N \times N$ jets with $\theta_0 = \pi/4$

N[a] / i	1		2		3		4		5	
	f_i	g_j	f_i	g_j	f_i	g_j	f_i	g_j	f_i	g_j
1	0	0	−0.122	−0.121	−0.192	−0.184	−0.237	−0.235	−0.269	−0.268
2	$J_1 = 0.1695$		0.122	0.121	0.008	0.008	−0.039	−0.039	−0.080	−0.080
3			$J_2 = 0.0766$		0.178	0.170	0.065	0.065	0.011	0.011
4					$J_3 = 0.0474$		0.196	0.194	0.108	0.108
5							$J_4 = 0.0292$		0.203	0.202
									$J_5 = 0.0235$	

[a] N varies from 1 to 5.

where $\phi(x,y)$ is the Cartesian equivalent of Eq. (25). It should be noted that $\overline{\Delta\phi^2}$ used in Eq. (1) is equal to $(3/4)^2\int_A \psi^2 \mathrm{d}x\mathrm{d}y$ divided by the area. Hence, $\overline{\Delta\phi_N^2} = (9/16)J_N/\pi$, where the area of a circular beam of normalized radius is π. The minimization of $J_N(f_i,g_j)$ was carried out by using a general purpose minimization program.[9] The results of these calculations for $\theta_0 = 0$ and $N = 1,2,\ldots,7$ are shown in Table 1 and for $\theta_0 = \pi/4$ and $N = 1,2,\ldots,5$ in Table 2. As was noted in Secs. II and III, the solution of this problem is not unique, and the data presented in Tables 1 and 2 are obtained by shifting the computed data so that Eq. (14) is satisfied. The main conclusion to be drawn from these calculations is about the rate of decrease of J_N with N, where J_N is a measure of the loss in the maximum intensity which can be achieved by focusing the laser beam. It is seen that most of the compensation is achieved by using a small matrix, say of 2×2 or 3×3 jets.

The main disadvantage of the solution just described is that it consumes a relatively large computer time. It was decided, therefore, to try the solution given by Eqs. (12-14) even though it is not strictly applicable, as just noted. In these equations ϕ_{ij} was replaced by $\overline{\phi_{ij}}$, where

$$\overline{\phi_{ij}} = \frac{1}{A_{ij}} \int_{A_{ij}} \phi(x,y)\,\mathrm{d}x\mathrm{d}y \tag{30}$$

The results are summarized in Table 3 for $\theta_0 = 0$ and in Table 4 for $\theta = \pi/4$. The agreement between the two methods is seen to be very good, and, since the second method is much faster and simpler than the first one, it seems to be more suitable for applications.

The continuous case, which corresponds to the limit $N \to \infty$, was solved by the iteration scheme defined by Eqs. (23) and (24) with the function $\phi(x,y)$ defined by Eqs. (25-28) and $\theta_0 = 0$. With an initial guess of $f^{(0)}(x) = g^{(0)}(y) = 0$, the process converged after three iterations. Again, the results have to be shifted in order to satisfy Eq. (19) and these

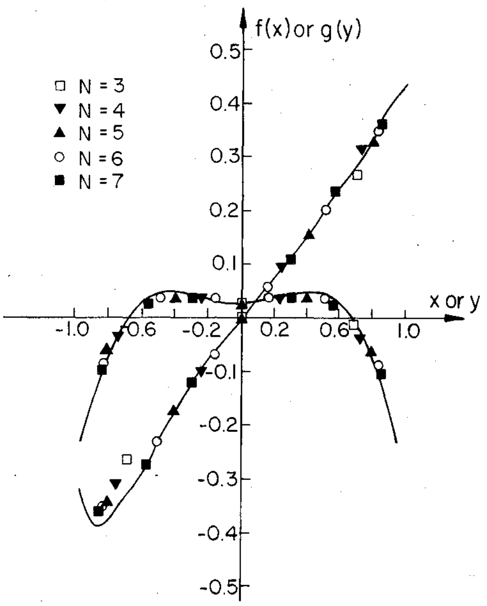

Fig. 5 Comparison between results for continuous jets and discrete jets for a laser beam with thermal blooming.

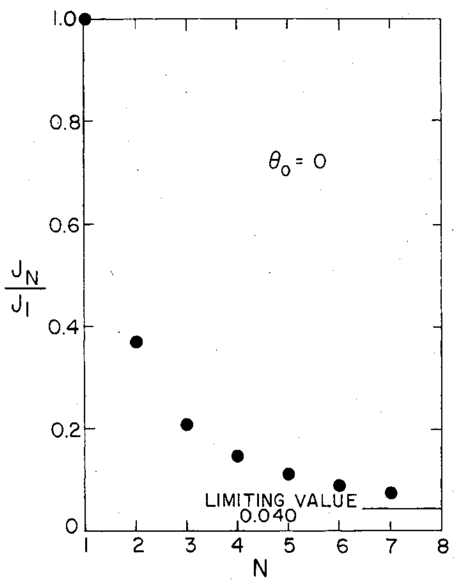

Fig. 6 Relative mean square phase compensation error vs the number of crossed jets; $\theta_0 = 0$.

solutions for $f(x)$ and $g(y)$ are displayed as the solid curves in Fig. 5. Also shown in Fig. 5 are the discrete solutions from Table 1, where f_i and g_j are located in the middle of the jets. The agreement between the continuous and the discrete solutions is seen to be good. Finally, in Fig. 6 the variation of J_N/J_1 as function of N is shown. The asymptotic value of $J_N/J_1 \rightarrow 0.040$ was computed by substituting the solutions $f(x)$ and $g(y)$ of the continuous case in Eq. (15).

The phase distortion has been expressed in terms of Zernike circle polynomials [Eq. (26)], each term of which corresponds to individual abberations such as tilt, focus, coma, and so forth.[7] In order to compare the results of those of Ref. 8, the functional J has been recomputed for the case of no tilt aberration [A_{11} set equal to zero in Eq. (26)]. Since the polynomials are orthonormal over the range of interest, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, this does not affect the other higher order aberrations and is equivalent to using a plane mirror to correct for tilt. These calculations were carried out and the results for J_N are given in Table 5 for $\theta_0 = 0$. The value of the rms phase error $\sqrt{\overline{\Delta\phi^2}} = \frac{3}{4}\sqrt{J_N/\pi}$, vs N is shown in Fig. 7 for

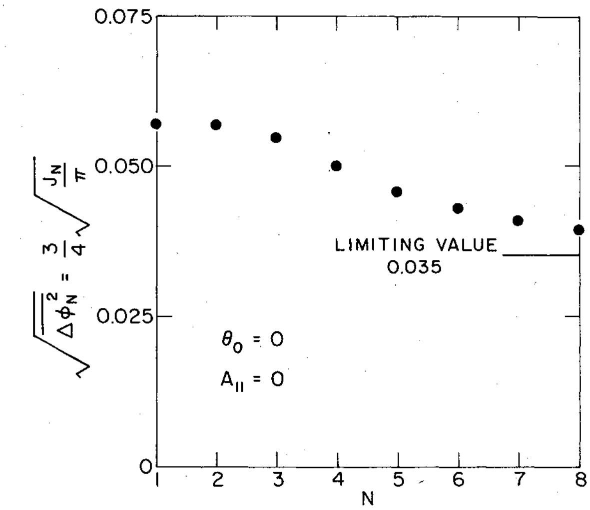

Fig. 7 rms phase compensation error vs the number of cross jets; tilt aberration set equal to zero, $\theta_0 = 0$.

Table 5 J_N vs $N \times N$ jets with tilt aberration and with both tilt and focus aberration set equal to zero for $\theta_0 = 0$

J_N \ N	1	2	3	4	5	6	7
$A_{11} = 0$	0.0181	0.0181	0.0166	0.0140	0.0118	0.0104	0.0095
$A_{11} = A_{20} = 0$	0.0152	0.0152	0.0138	0.0123	0.0112	0.00972	0.00888

$\theta_0 = 0$. It is seen that the relative improvements, while substantial, are not as good as those which include tilt. In fact, for a small matrix of 2×2 jets, the mean square of the error is not decreased at all. This is because only a tilt correction is possible with a small number of jets. Figure 7 may be used to compare with Fig. 9 in Ref. 8, where a similar effect of reducing the phase error is achieved by deforming the mirror. By dividing the rms phase compensation error shown in Fig. 9 of Ref. 8 by $\lambda/2\pi = 10.6\mu/2\pi$, the ordinates of the two figures are made identical. As the number of actuators in the freejet system is equivalent to $2N$, one can compare the relative improvement between Ref. 8 and this work on a point by point basis. For example, the rms phase error for $N = 6$ can be compared with point 6 in Ref. 8 as both have twelve actuators. The rms phase error for point 6 is 0.027 whereas that for the freejet system is 0.042. Thus the best twelve-actuator geometry shown in Ref. 8 (i.e., point 6) gives a smaller error than a 6×6 array of jets. While the rms phase error is reduced by 27% for $N = 6$, increasing the number of jets further is not too effective as the limiting value of $\sqrt{\overline{\Delta\phi^2}}$ is 0.035. Hence, the difference in phase error between the freejet system and the deformable mirror system becomes more marked as the number of actuators increases. If further reduction in phase error is desired using freejets, it would be worthwhile to investigate the effects of adding additional jets in different directions.

A practical adaptive optics system may be one in which tilt and focus can both be corrected by the transmitter itself. In practice, this may be done by simply pointing the laser in a slightly different direction and altering its focusing optical elements. A similar calculation without tilt and focus ($A_{11} = A_{20}$ set equal to zero) has been done, and results for this case also are given in Table 5. As can be seen, there is not much difference with the previous case. It may be noted that the jets in principle handle tilt very well, as can be seen by the rapid decrease in the variation in J_N with N in Fig. 6. Although tilt may be handled by the transmitter for steady or slowly varying tilt errors, pointing jitter or rapid variations in tilt due to atmosphere turbulence may be advantageously handled by the jet system itself.

V. Conclusion

The possibility of compensation for phase error of a laser beam by ideal gas jets which are interposed in the beam path has been studied for distortion due to thermal blooming, and it was found that only tilt was correctable with a small matrix of jets. Using a larger number of equal width jets, however, the mean square of the phase error ignoring tilt could be reduced by a factor of approximately 50%. Increasing the number of jets beyond a 7×7 matrix is not effective, and if further reduction is desired it is better to insert new jets in different directions.

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